

SHAPE QUANTITIES FOR RELATIONAL QUADRILATERALLAND

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Abstract

I investigate useful shape quantities for the classical and quantum mechanics of the relational quadrilateral in 2-d. This is relational in the sense that only relative times, relative ratios of separations and relative angles are significant. Relational particle mechanics models such as this paper's have many analogies with the geometrodynamical formulation of general relativity. This renders them suitable as toy models for 1) studying Problem of Time in Quantum Gravity strategies, in particular timeless, semiclassical and histories theory approaches and combinations of these. 2) For consideration of various other quantum-cosmological issues, such as structure formation/inhomogeneity and notions of uniform states and their significance. The relational quadrilateral is more useful in these respects than previously investigated simpler RPM's due to simultaneously possessing linear constraints, nontrivial subsystems and nontrivial complex-projective mathematics. Such shape have been found to be useful in simpler relational models such as the relational triangle and in 1-d.

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1 Introduction

Relational particle mechanics (RPM) are mechanics in which only relative times, relative angles and (ratios of) relative separations are physically meaningful. Scaled RPM and pure-shape (i.e. scalefree, and so involving just ratios) RPM were set up respectively in [1] by Barbour–Bertotti and in [2] by Barbour. These theories are relational in Barbour’s sense rather than Rovelli’s [3, 4, 5, 6], via obeying the following postulates. Firstly, RPM’s are *temporally relational* [1, 7, 8], i.e. there is no meaningful primary notion of time for the whole (model) universe. This is implemented by using actions that are manifestly reparametrization invariant while also being free of extraneous time-related variables [such as Newtonian time or the lapse in General Relativity (GR)]. This reparametrization invariance then directly produces primary constraints quadratic in the momenta (which in the GR case is the super-Hamiltonian constraint, and for RPM’s is an energy constraint.) Secondly, RPM’s are *configurationally relational*, which can be thought of in terms of a certain group G of transformations that act on the theory’s configuration space Q being physically meaningless [1, 7, 8, 9, 10]. Configurational relationalism can be approached by [1, 2] indirect means that amount to working on the principal bundle $P(Q, G)$. Then one has linear constraints from varying with respect to the G -generators (the super-momentum constraint in the case of GR, zero total angular momentum for the universe for scaled RPM and both that, and its dot- rather than cross-product counterpart, zero total dilational momentum, for pure-shape RPM). Then elimination of the G -generator variables from the Lagrangian form of these constraints sends one to the desired quotient space¹ Q/G . In 1- and 2- d , these RPM’s can be obtained alternatively by working directly on Q/G [8, 10]. This involves applying a ‘Jacobi–Synge’ construction of the natural mechanics associated with a geometry, to Kendall’s shape spaces [11] for pure-shape RPM’s or to the cones over these [10] for the scaled RPM’s.

RPM’s are furthermore motivated through being useful toy models [12, 5, 7, 13, 14, 9, 15, 10] of GR in geometrodynamical form, since they possess analogues of many features of this. RPM’s are similar in complexity and in number of parallels with GR to minisuperspace [17] and to 2 + 1 GR [18] are, though each such model differs in how it resembles GR, so that such models offer complementary perspectives and insights. In particular, RPM’s purely quadratic energy constraint parallels the GR super-Hamiltonian constraint in leading to the frozen formalism facet of the Problem of Time [12, 19], alongside nontrivial linear constraints that parallel the GR momentum constraint (and are absent for minisuperspace), which cause a number of further difficulties in various approaches to the Problem of Time. Other useful applications of RPM’s not covered by minisuperspace models include that RPM’s are useful for the qualitative study of the quantum-cosmological origin of structure formation/inhomogeneity (scaled RPM’s are a tightly analogous, simpler version of Halliwell–Hawking’s [20] model for this), and for the study of correlations between localized subsystems of a given instant. Scalefree RPM’s such as this paper’s, moreover, occurs as subproblems within scaled RPM’s, corresponding to the light fast modes/inhomogeneities of shape in the preceding sentence’s context. RPM’s also admit analogies with conformal/York-type initial value problem formulations of GR, with the conformal 3-geometries playing an analogous role to the pure shapes [16]. In general, having multiple toy models for the Problem of Time/Quantum Cosmology is highly useful [12], and developing RPM’s adds to this.

1- d RPM’s [15, 21] and triangle-land in 2- d [6, 22, 23, 24] have already been covered. 2- d suffices for almost all the analogies with GR to hold whilst keeping the mathematics manageable: for 1- and 2- d pure-shape RPM’s the reduced configuration spaces (shape spaces) are \mathbb{S}^{N-2} and \mathbb{CP}^{N-2} ; for scaled RPM’s, they are the cones over these. Thus, we are now looking at the case of quadrilateral-land in 2- d . Quadrilateral-land is valuable A) through its simultaneously possessing nontrivial subsystems and nontrivial constraints, rendering it a useful and nontrivial model from which to gain qualitative understanding of the Halliwell–Hawking model and of correlations within a single instant, which enter into many timeless approaches to the Problem of Time (e.g. [33, 14] and more references in [30]). B) From the 2- d RPM of N particles having \mathbb{CP}^{N-2} mathematics, of which $N = 4$ is the first genuinely nontrivial and thus typical case (since, for $N = 3$, $\mathbb{CP}^1 = \mathbb{S}^2$). I further lay out mathematics of the quadrilateral in 2- d in Sec 2, and discuss some of the consequences and applications in the Conclusion.

Shape quantities are useful [22, 15, 23, 10, 21, 24] in interpreting the classical and quantum solutions of RPM’s and for the kinematical quantization [25] of RPM’s [15, 24]. For the relational triangle, a useful set of such are Dragt-type coordinates [26]; these are nontrivially realized Cartesian coordinates associated with a \mathbb{R}^3 surrounding the \mathbb{S}^2 shape space. These can furthermore be interpreted as an ellipticity of the moments of inertia, a departure from isoscelesness (‘anisoscelesness’) and four times the area per unit moment of inertia of the (mass-weighted) triangle ([23] and Sec 4). Ellipticity and anisoscelesness depend on how one splits the subsystem up into clusters, while the area is a clustering-independent property, alias *democracy invariant* [27, 28, 29] (Sec 3). The current paper addresses the question of what the relational quadrilateral’s counterparts of these Dragt-type quantities are (Sec 5). I find that they are 3 anisoscelesnesses, 1 ellipticity, 1 linear combination of 2 ellipticities and a quantity proportional to the square root of the sum of squares of areas. (All of these quantities refer to Sec 2’s constituent subsystem triangles present within the quadrilateral.) Among these, the quantity related to the areas is the sole democracy invariant shape quantity. The current paper has useful applications to 1) investigations of the classical and quantum mechanics of the relational quadrilateral (paralleling those in [6, 22, 23, 24] for the relational triangle). 2) For subsequent investigations of Problem of Time in Quantum Gravity

¹Here, Q is the naïve configuration spaces for N particles in dimension d , \mathbb{R}^{Nd} . G is a group of transformations that are declared to be physically meaningless. E.g. 1) In the case of scaled RPM, the Euclidean group of translations and rotations (removing what is usually regarded as the absolutist content of mechanics). E.g. 2) the similarity group that furthermore includes dilations in the case of pure-shape RPM.

strategies [30, ?] and for investigation of various other conceptual issues in Quantum Cosmology [33, 15, 23, 39, 32], including in particular the role of democratic invariants as diagnostics of uniform states. Uniform states play an important role in both Classical and Quantum Cosmology, since the current universe is fairly uniform and initial conditions from which it arose are usually taken to be extremely uniform.

2 The relational quadrilateral

N particles in 2- d has² Cartesian configuration space \mathbb{R}^{2N} . Rendering absolute position irrelevant (e.g. by passing from particle position coordinates³ \mathbf{q}^a to any sort of relative coordinates) leaves one on the configuration space *relative space*, $\mathbf{R} = \mathbb{R}^{2n}$, for $n = N - 1$. The most convenient sort of coordinates for this are *relative Jacobi coordinates* [34] \mathbf{R}^e . These are combinations of relative position vectors $\mathbf{r}^{ab} = \mathbf{q}^b - \mathbf{q}^a$ between particles into inter-particle cluster vectors such that the kinetic term is diagonal. E.g. for 3 particles, these are $\mathbf{R}_1 = \mathbf{q}_3 - \mathbf{q}_2$ and $\mathbf{R}_2 = \mathbf{q}_1 - \{m_2\mathbf{q}_2 + m_3\mathbf{q}_3\}/\{m_2 + m_3\}$. Relative Jacobi coordinates have associated particle cluster masses μ_e . In fact, it is tidier to use *mass-weighted* relative Jacobi coordinates $\boldsymbol{\rho}^e = \sqrt{\mu_e}\mathbf{R}^e$ (Fig 1) or the squares of their magnitudes: partial moments of inertia $I^e = \mu_e|\mathbf{R}^e|^2$, or the normalized counterparts of the former, $\mathbf{n}^e = \boldsymbol{\rho}^e/\rho$. Here, $\rho = \sqrt{I}$ is the *hyperradius*, I itself being the moment of inertia of the system. I use (Hb) as shorthand for {ab, cd} i.e. the clustering (partition into subclusters) into two pairs ab and cd, and (Ka) as shorthand for {{bc, d}, a} i.e. the clustering into a single particle a and a triple {bc, d} which is itself partitioned into a pair bc and a single particle d. In each case, a, b, c, d form a cycle. I take clockwise and anticlockwise labelled triangles to be distinct, i.e. I make the plain rather than mirror-image-identified choice of set of shapes. I also assume equal masses for simplicity.

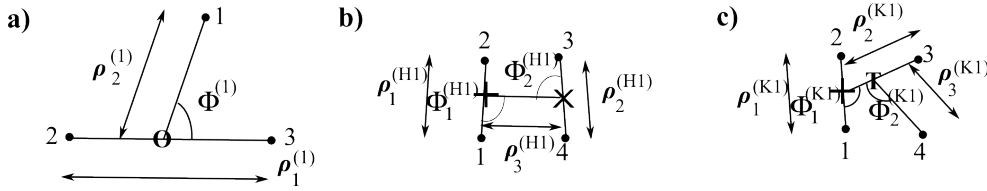


Figure 1: O, + and \times denote COM(23), COM(12) and COM(34) respectively, where COM(ab) is the centre of mass of particles a and b. a) For 3 particles in the plane, one particular choice of mass-weighted relative Jacobi coordinates are as indicated. I furthermore define $\Phi^{(a)}$ as the ‘Swiss army knife’ angle $\arccos(\rho_1^{(a)} \cdot \rho_3^{(a)} / \rho_1^{(a)} \rho_3^{(a)})$. b) For 4 particles in the plane, one particular choice of mass-weighted relative Jacobi H-coordinates are as indicated. I furthermore define $\Phi_1^{(Hb)}$ and $\Phi_2^{(Hb)}$ as the ‘Swiss army knife’ angles $\arccos(\rho_1^{(Hb)} \cdot \rho_3^{(Hb)} / \rho_1^{(Hb)} \rho_3^{(Hb)})$ and $\arccos(\rho_2^{(Hb)} \cdot \rho_4^{(Hb)} / \rho_2^{(Hb)} \rho_4^{(Hb)})$ respectively. c) For 4 particles in the plane, one particular choice of mass-weighted relative Jacobi K-coordinates are as indicated. I furthermore define $\Phi_1^{(Ka)}$ and $\Phi_2^{(Ka)}$ as the ‘Swiss army knife’ angles $\arccos(\rho_1^{(Ka)} \cdot \rho_2^{(Ka)} / \rho_1^{(Ka)} \rho_2^{(Ka)})$ and $\arccos(\rho_2^{(Ka)} \cdot \rho_3^{(Ka)} / \rho_2^{(Ka)} \rho_3^{(Ka)})$ respectively.

If one takes out the rotations also, one is on *relational space*. If one furthermore takes out the scalings, one is on *shape space*. If one takes out the scalings but not the rotations, one is on *preshape space* [11]. For N particles in 2- d , preshape space is \mathbb{S}^{nd-1} and shape space is \mathbb{CP}^{N-2} (provided that the plain choice of set of shapes is made). The 3-particle case of this is, moreover, special, by $\mathbb{CP}^1 = \mathbb{S}^2$. Now, relational space is *the cone* [10] over shape space, which is $C(\mathbb{CP}^{N-2})$ in the present case [and $C(\mathbb{CP}^1) = C(\mathbb{S}^2) = \mathbb{R}^3$, albeit not with the flat metric upon it [24]]. Thus, we consider this paper’s quadrilateralland case as the first case with nontrivial complex-projective maths.

In this scheme, the kinetic metric on shape space is the natural Fubini-Study metric [11, 8],

$$ds_{\text{FS}}^2 = ((1 + |Z|_c^2)|dZ|_c^2 - |(Z, d\bar{Z})_c|^2)/(1 + |Z|_c^2)^2. \quad (1)$$

Here, $|Z|_c^2 = \sum_I |Z^I|^2$, $(\cdot, \cdot)_c$ is the corresponding inner product, Z^I are the inhomogeneous coordinates [independent set of ratios of $z^e = \mathcal{R}^e \exp(i\Theta^e)$ for \mathcal{R}^e , Θ^e the polar presentation of Jacobi coordinates], the overline denotes complex conjugate and $|\cdot|$ the complex modulus. Note that using $Z^I = \mathcal{R}^I \exp(i\Theta^I)$, the corresponding kinetic term can be recast in real form. The kinetic metric on relational space in scale-shape split coordinates is then

$$ds_{\text{C(FS)}}^2 = d\rho^2 + \rho^2 ds_{\text{FS}}^2. \quad (2)$$

The action for N -gonland in relational form is then

$$2 \int d\lambda \sqrt{\mathcal{T}\{U + E\}}. \quad (3)$$

This is a Jacobi-type action [35] (thus complying with temporal relationalism), with $\mathcal{T} = \mathcal{T}_{\text{FS}}$ or $\mathcal{T}_{\text{C(FS)}}$ built from ds_{FS}^2 or $ds_{\text{C(FS)}}^2$ respectively (thus directly implementing configurational relationalism).

²I refer to this as N -a-gonland, the first two nontrivial subcases of which are triangleland ($N = 3$) and quadrilateralland ($N = 4$).

³I take a, b, c as particle label indices running from 1 to N for particle positions. e, f, g as particle label indices running from 1 to $n = N - 1$ for relative position variables. I, J, K as indices running from 1 to $n - 1$. i, j, k as spatial indices. Δ as an index running over the $SO(N)$ indices (1 to 3 for triangleland and 1 to 6 for quadrilateralland). δ as an index running over all bar one of the $SO(N)$ indices. p, q, r as relational space indices (running from 1 to 3 for triangleland and from 1 to 5 for quadrilateralland). u, v, w as shape space indices (running from 1 to 4 for quadrilateralland).

3 Democracy transformations for 1- and 2-d RPM's

The *democracy transformations* [27, 28, 29] (alias kinematical rotations [36])

$$\boldsymbol{\rho}_i \longrightarrow \boldsymbol{\rho}'_i = D_{ij}\boldsymbol{\rho}_j \quad (4)$$

are interpolations between relative Jacobi vectors for different choices of clustering. These transformations form the *democracy group* $\text{Demo}(n)$, which is $O(n)$, or, without much loss of generality, $SO(n)$. Note that this is a symmetry group of the unreduced kinetic term. Also note that the democracy group is indeed independent of spatial dimension, since democracy transformations only mix up whole Jacobi vectors rather than separately mixing up each's individual components. This means, firstly, that much about the theory of democracy transformations in 2- d can be gained from their more widely studied counterpart in 3- d . Secondly, it means that for spatial dimension > 1 , the democracy group is but a subgroup of the full symmetry group of the unreduced kinetic term, which itself is the dimension-dependent $O(nd)$.

Lemma 1. The hyperradius ρ is always a democracy invariant. It is, however, a scale rather than shape quantity.

Lemma 2. $\dim(\mathcal{R}(n, d)) - \dim(\text{Demo}(n)) = (n + d - (n - d)^2)/2$ gives a minimum number of independent democracy invariants [28]. This bound is 2 both for triangleland and for quadrilateralland, revealing in each case the presence of at least one further democracy invariant.

It is intuitively clear that one such for triangleland should be the area of the triangle. It is not, however, clear what happens for quadrilateralland, due to RPM's in truth involving constellations, i.e. sets of points rather than figures formed from these points. Then, for $N > 3$, there is an ambiguity in how one would 'join up the dots'. Thus the area ascribed to the constellation via drawing a quadrilateral between its points is not itself a democracy invariant.

To find democracy invariants, consider the independent invariants that arise from the Q -matrix [28]

$$Q_{ij} = \boldsymbol{\rho}_i \boldsymbol{\rho}_j^T. \quad (5)$$

The invariant $\text{trace}(Q)$ always gives the hyperradius, hence proving Lemma 1. For N -stop metroland, $Q_{ij} = \rho_i \rho_j$, which is sufficiently degenerate that it and all of its nontrivial submatrices have zero determinants, so that there are no more democracy invariants. For triangleland, however, $\det(Q) = 4 \text{Area}^2$ (for Area the area of the triangle per unit I in mass-weighted space), while Q_{ij} is but 2×2 , so that there are no more invariants. For quadrilateralland, $\det(Q) = 0$. This is understandable in terms of volume forms being zero for planar figures. Also, being 3×3 , there is one further invariant, $Q_{11}Q_{22} - Q_{12}^2 + \text{cycles} = \rho_1^2 \rho_2^2 - \{\boldsymbol{\rho}_1 \cdot \boldsymbol{\rho}_2\}^2 + \text{cycles} = |\boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2|_3^2 + \text{cycles}$, i.e. proportional to the sum of squares of areas of various constituent triangles (see Fig 2). Here the 3-suffix denotes that it is mathematically the component in a fictitious third dimension of the cross product. In the $\{12, 34\}$ H-clustering, these are the coarse-grainings by taking 1, 2 and COM(34); 3, 4 and COM(12); 12 and 34 with one shifted so that its COM coincides with the other's. In the $\{\{12, 3\}, 4\}$ K-clustering, these are 12 and 3 with 4 at COM(123); 3, 4 and COM(12); 12, 4 with particle 3 at COM(12). These triangles clearly correspond to those that one makes by taking 2 out of the 3 relative Jacobi variables; as such, they play further roles below. This is also indicative of how quadrilateralland has a role for nontrivial subsystems, which is of interest especially as regards timeless approaches (see the Conclusion for more).

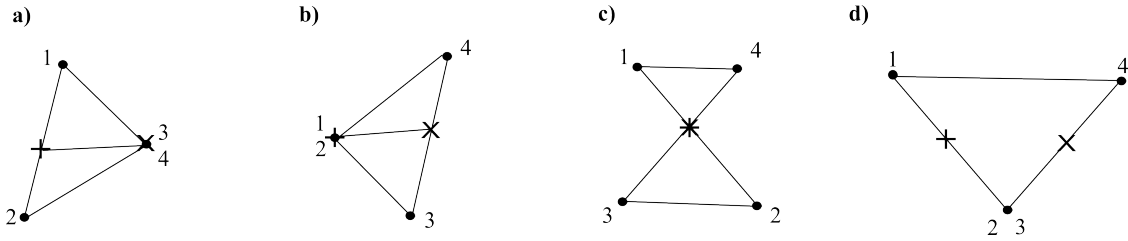


Figure 2: Figure of the constituent triangles for the case of (H2)-coordinates, i.e. those made from each pair of $\rho^{(H2)}$ -vectors. The (31) triangle (made from the $\rho_1^{(H2)}$ and $\rho_3^{(H2)}$ vectors) a) and the (23) triangle (made from the $\rho_2^{(H2)}$ and $\rho_3^{(H2)}$ vectors) b) are straightforward, whilst the (12) case c) requires a displacement in order to form a triangle out of the $\rho_1^{(H2)}$ and $\rho_2^{(H2)}$ vectors [d)].

4 Shape quantities for triangleland

I present a different method of finding the useful shape coordinates for triangleland, which continues to work in the quadrilateralland case, as well as providing a useful prequel for this as regards notation and interpretation. \underline{Q} may be written as $\frac{1}{2}(w_1 \underline{\sigma}_1 + w_2 \underline{\sigma}_2 + w_3 \underline{\sigma}_3)$ (for Pauli matrices $\underline{\sigma}_1$ and $\underline{\sigma}_3$; $\underline{\sigma}_2$ does not feature since \underline{Q} is symmetric). This takes such an $SU(2)$ form due to the 3-particle case being exceptional through the various tensor fields of interest possessing higher symmetry in this case [28]. Then, reading off from the definition of \underline{Q} [and dropping (1)-labels],

$$w = \rho^2 = I(= \text{moment of inertia}) := \text{Size} , \quad (6)$$

$$w_1 = \rho_1^2 - \rho_2^2 := \text{Ellip} , \quad (7)$$

$$w_3 = 2 \boldsymbol{\rho}_1 \cdot \boldsymbol{\rho}_2 := \text{Aniso} . \quad (8)$$

w_1 can be interpreted as an ‘ellipticity’, being the difference in the principal moments of inertia of the base and median. Also, w_3 can be interpreted as an ‘anisoscelesness’ (i.e. departure from isoscelesness, in analogy with anisotropy as a departure from isotropy in GR Cosmology); specifically, working in mass-weighted space, Aniso per unit base length is the amount by which the perpendicular to the base fails to bisect it ($l_1 - l_2$ from Fig 3).

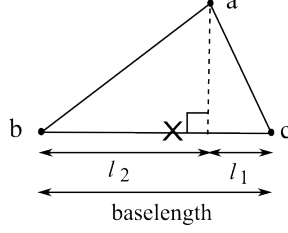


Figure 3: Definitions of l_1 and l_2 used in the anisoscelesness notion Aniso(a).

There is also a

$$w_2 = 2 |\boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2|_3 \quad (9)$$

such that $w^2 = \sum_{i=1}^3 w_i^2$. (9) has straightforward interpretation as $4 \times \text{Area}$, which is a measure of noncollinearity and, clearly, a democratic invariant notion.

For triangleland, w_1 , w_2 and w_3 are Dragt-type coordinates. [Though, given that these run over not Dragt’s \mathbb{R}_+^3 but \mathbb{R}^3 , they are even more closely related to the well-known $\mathbb{S}^3 \rightarrow \mathbb{S}^2$ Hopf Map. This is involved in the configuration space study as follows: relative space $\mathbb{R}^4 \rightarrow$ preshape space $\mathbb{S}^3 \xrightarrow{\text{Hopf map}} \text{shape space } \mathbb{S}^2 = \mathbb{CP}^1 \xrightarrow{\text{coning}} \mathbb{R}^3 \text{ relational space.}]$ For these to be shape quantities, I divide each of these by I and switch the 2 and 3 labels (so the principal axis is aligned with the democracy invariant), obtaining

$$\text{shape}_1 = n_1^2 - n_2^2 = \text{Ellip}/I := \text{ellip} , \quad (10)$$

$$\text{shape}_2 = 2 \mathbf{n}_1 \cdot \mathbf{n}_2 = \text{Aniso}/I := \text{aniso} , \quad (11)$$

$$\text{shape}_3 = 2 |\mathbf{n}_1 \times \mathbf{n}_2|_3 = 4 \times \text{Area}/I := 4 \times \text{area} := \text{demo}(N=3) . \quad (12)$$

These are in $\mathbb{R}^3 = C(\mathbb{S}^2)$ and subject to the on- \mathbb{S}^2 restriction $\sum_{\Delta=1}^3 \text{shape}_{\Delta}^2 = 1$. Interpretation of various great circles and of the hemispheres they divide triangleland into are provided in Fig 4.

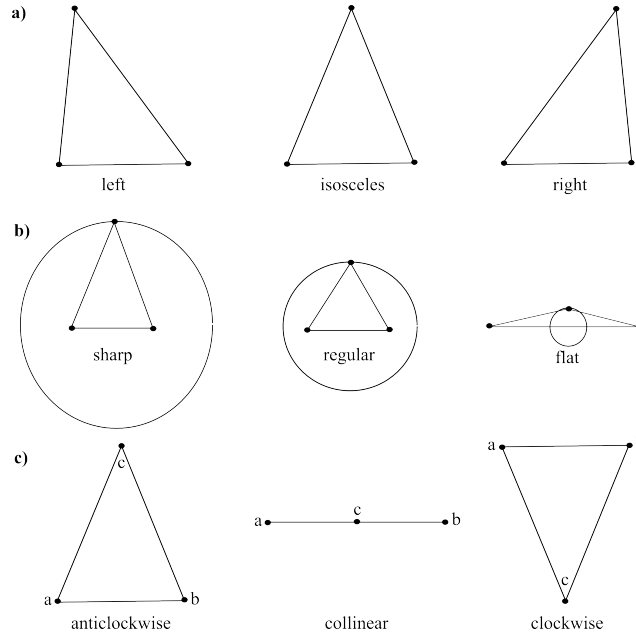


Figure 4: a) Each bimeridian of isoscelesness cuts the shape space sphere into hemispheres of right and left triangles. b) Each bimeridian of regularness ($I_1 = I_2$) cuts the shape space sphere into hemispheres of sharp ($I_1 < I_2$) and flat triangles ($I_1 > I_2$). c) The equator of collinearity cuts the shape space sphere into hemispheres of anticlockwise and clockwise triangles.

5 Shape quantities for quadrilateralland

Repeating this for quadrilateralland closely parallels the analysis for the tetrahaedron in [28] because the democracy group does not care about dimension. Now,

$$\underline{\underline{Q}} = \frac{1}{2} \left\{ w \underline{\underline{1}} + \sum_{\delta=1}^5 w_{\delta} \underline{\underline{B}}_{\delta} \right\}, \quad (13)$$

where the $\underline{\underline{B}}_{\Delta}$ are $j=1$ [quantum number of the $SO(3)$ democracy group] representation matrices,

$$\underline{\underline{B}}_1 = \sqrt{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \underline{\underline{B}}_2 = \sqrt{3} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \underline{\underline{B}}_3 = \sqrt{3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \underline{\underline{B}}_4 = \sqrt{3} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \underline{\underline{B}}_5 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \quad (14)$$

Thus, reading off from the definition of $\underline{\underline{Q}}$ and dropping (H2) labels,

$$w = Q_{11} + Q_{22} + Q_{33} = \rho_1^2 + \rho_2^2 + \rho_3^2 = \rho^2 = I : \text{moment of inertia (MOI)}, \quad (15)$$

$$w_1 = \sqrt{3}(Q_{11} - Q_{22})/2 = \sqrt{3}(\rho_1^2 - \rho_2^2)/2 = \sqrt{3} \text{Ellip}(12)/2, \quad (16)$$

$$w_2 = \sqrt{3}Q_{12} = \sqrt{3}\boldsymbol{\rho}_1 \cdot \boldsymbol{\rho}_2 = \sqrt{3} \text{Aniso}(12), \quad (17)$$

$$w_3 = \sqrt{3}Q_{23} = \sqrt{3}\boldsymbol{\rho}_2 \cdot \boldsymbol{\rho}_3 = \sqrt{3} \text{Aniso}(23), \quad (18)$$

$$w_4 = \sqrt{3}Q_{31} = \sqrt{3}\boldsymbol{\rho}_3 \cdot \boldsymbol{\rho}_1 = \sqrt{3} \text{Aniso}(31), \quad (19)$$

$$w_5 = (-Q_{11} - Q_{22} + 2Q_{33})/2 = (-\rho_1^2 - \rho_2^2 + 2\rho_3^2)/2 = (\text{Ellip}(31) + \text{Ellip}(32))/2. \quad (20)$$

Here, the brackets refer to the pair of Jacobi vectors involved in each constituent triangle (Fig 2); note that Ellip being a signed quantity requires this to be an ordered pair.

Note also that if one sets

$$w_6 = \sqrt{3}\{|\boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2|_3^2 + \text{cycles}\}^{1/2} = \sqrt{12}\{\text{Area}(12)^2 + \text{cycles}\}^{1/2}, \quad (21)$$

i.e. a quantity proportional to the square root of the above-found democracy invariant, then $w^2 = \sum_{\Delta=1}^6 w_{\Delta}^2$. One can then straightforwardly make the w_{Δ} into shape quantities by dividing through by I :

$$\text{shape}_1 := \sqrt{3}(\mathbf{n}_1^2 - \mathbf{n}_2^2)/2 = \sqrt{3} \text{ellip}(12)/2 \quad (22)$$

$$\text{shape}_2 := \sqrt{3} \mathbf{n}_1 \cdot \mathbf{n}_2 = \sqrt{3} \text{aniso}(12), \quad (23)$$

$$\text{shape}_3 := \sqrt{3} \mathbf{n}_2 \cdot \mathbf{n}_3 = \sqrt{3} \text{aniso}(23), \quad (24)$$

$$\text{shape}_4 := \sqrt{3} \mathbf{n}_3 \cdot \mathbf{n}_1 = \sqrt{3} \text{aniso}(31), \quad (25)$$

$$\text{shape}_5 := (-\mathbf{n}_1^2 - \mathbf{n}_2^2 + 2\mathbf{n}_3^2)/2 = (\text{ellip}(31) + \text{ellip}(32))/2, \quad (26)$$

$$\text{shape}_6 := \sqrt{3}(|\mathbf{n}_1 \times \mathbf{n}_2|_3^2 + \text{cycles})^{1/2} = \sqrt{12}(\text{area}(12)^2 + \text{cycles})^{1/2} = \text{demo}(N=4). \quad (27)$$

These belong to $\mathbb{R}^6 = \mathbb{C}^3$ and the relation between the w^{Δ} becomes the on- \mathbb{S}^5 condition $\sum_{\Delta=1}^6 \text{shape}_{\Delta}^2 = 1$. That an \mathbb{S}^5 makes an appearance is not unexpected (for quadrilateralland, from Sec 2 and using the generalization of the Hopf map in the sense of $U(1) = SO(2)$ fibration, relative space $\mathbb{R}^6 \rightarrow \text{preshape space } \mathbb{S}^5 \xrightarrow{\text{Hopf map}} \text{shape space } \mathbb{CP}^2$). Moreover, since \mathbb{CP}^2 is 4- d and \mathbb{S}^5 is 5- d , there is a further condition on the shape_{Δ} for quadrilateralland. Also above, $\text{aniso}(12)$ is same notion of anisoscelesness as before but now applied to the triangle made out of the 1 and 2 relative Jacobi vectors, and so on. $\text{demo}(N=4) = \sqrt{12}$ times squares of mass-weighted areas per unit MOI, which is a democracy invariant.

Quadrilateralland's relational space is locally \mathbb{R}^5 . One can envisage this from $\mathbb{CP}^2/\mathbb{Z}_2 = \mathbb{S}^4$ [38] and then taking the cone. As we are treating \mathbb{CP}^2 , we have, rather, two copies of \mathbb{S}^4 . Such a doubling does not appear in the coplanar boundary of [28]'s study of 4 particles in 3- d , since 3- d dictates that the space of shapes on its coplanar boundary is the mirror-image-identified one. Another distinctive feature of 3- d is that it has a third volume-type democracy invariant, $\boldsymbol{\rho}_1 \cdot \boldsymbol{\rho}_2 \times \boldsymbol{\rho}_3$.

Finally $\text{demo}(N=4) = \text{shape}_6$ can be used to diagnose collinearity as the points at which this is zero. It is in this respect like triangleland's $\text{demo}(N=3) = \text{shape}_3$, which also gives the equilateral configurations (corresponding to a manifest notion of maximum uniformity) as the output to its extremization, corresponding to the natural \mathbb{S}^2 poles. However, one can quickly see that extremizing $\text{demo}(N=4)$ will involve something more complicated, since there are not two but six permutations of labelled squares (likewise a manifest notion of maximum uniformity), so these cannot possibly all lie on the two \mathbb{S}^5 poles, nor could any of them lie on the poles by a symmetry argument. In fact, extremizing $\text{demo}(N=4)$ does yield these squares, but not uniquely. One gets, rather, one curve of extrema for each of the two orientations, each containing three labelled squares.

6 Conclusion

In this paper, I showed how to extend the relational triangle’s useful shape quantities (ellipticity, anisoclesness and four times the area) to the case of the relational quadrilateral. We get one ellipticity, one linear combination of ellipticities, three anisoclesnesses and a quantity proportional to the square root of the sum of three areas squared. These quantities refer to three cluster-dependent ‘subsystem triangles’ (Fig 2). The last of these quantities, like the triangle case’s $4 \times \text{area} [= \text{demo}(N = 3)]$, is furthermore a democratic invariant (cluster independent quantity), $\text{demo}(N = 4)$.

In $2-d$, the extension to the general ‘ N -a-gonland’ is neat because these have \mathbb{CP}^{N-2} as their shape spaces; this is in contrast to the $3-d$ models’ shape spaces forming a much harder sequence [11], which is a good reason to study the $2-d$ models (particularly since the analogy with geometrodynamics does not require dimension 3 in order to work well). However, now the number of independent ellipticities plus the one hyperradius gives n , there continues to be a sum of squares of areas type of democracy invariant, but, additionally, the number of anisoclesnesses goes as $n(n - 1)/2$, so that the latter substantially overwhelms the shape space dimension. This suggests that the these more general models will require a tighter set of shape variables than the present paper’s one which includes all of its possible anisoclesness quantities. I leave this (and its application to kinematical quantization) as questions for a further occasion, since the present paper’s coverage of the quadrilateralland case presently suffices to advance my research program.

This paper’s shape quantities are useful in the following investigations of the classical and quantum mechanics of the relational quadrilateral [39, 32] (paralleling those in [6, 22, 23, 24] for the relational triangle, including expectations and spreads of corresponding operators).

- A) I look to study of special configurations and subregions of quadrilateralland’s shape space that they represent.
- B) I physically interpret quadrilateralland’s conserved quantities (these form the group $SU(3)$ as opposed to the $SU(2)$ or $SO(3)$ form of triangleand’s and 4 particles on a line’s [15, 23]). I then aim to treat quadrilateralland’s classical equations of motion and Schrödinger equation.
- C) For subsequent investigations of Problem of Time in Quantum Gravity strategies and various other quantum-cosmological issues. Particular such application for Quadrilateralland and N -a-gonland are to
- I) timeless approaches to the Problem of Time in Quantum Gravity,
- II) qualitative models of structure formation in Quantum Cosmology, and
- III) the robustness study based on the $(N - 1)$ -particle model lying inside the N -particle one. I note that complex projective mathematics (the present paper being the first specific RPM model paper that involves nontrivial such) will underly this robustness study.

The large- N limit of N -a-gonland may also be interesting, alongside the study of the statistical mechanics/entropy/notions of information/relative information/correlation that timeless approaches are concerned with. The present paper’s model is also the smallest that possesses both linear rotational constraints (conferring it a ‘midisuperspace toy model character’) and nontrivial subsystems (applications of which include attempting to use one such as a clock for the other parts of the model universe [37, 3], timeless records approaches [14] and structure formation.)

IV) Another application is to notions of uniformity and how probable such states are (this is another issue of interest in Quantum Cosmology). Now, while $\text{demo}(N = 4)$ for quadrilateralland is as good a diagnostic as $\text{demo}(N = 3)$ for triangleand as regards collinear configurations, that the extrema of $\text{demo}(N = 3)$ pick out the triangleand uniform states (the 2 labellings of equilateral triangle) only partly carries over to the extrema of $\text{demo}(N = 4)$, since these do not uniquely pick out the quadrilateral uniform states (the 6 labellings of squares). Obtaining more information of where these uniform states sit in quadrilateralland’s \mathbb{CP}^2 configuration space (with or without orientation and distinguishability reducing these to 3 or just 1 uniform state) remains work in progress [39].

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